E9 203 Compressed Sensing & Sparse SP

14 May 2021.

Last time:

Thm. Let $0 < \delta < 1$, $A \in \mathbb{R}^{m \times N}$, $A_{ij} \overset{iid}{\sim} \mathcal{N}(0, \frac{1}{m})$.
If $m \geq \kappa_1 k \log\left(\frac{N}{k}\right)$, then A satisfies RIP of order k with the prescribed δ with probability exceeding $\left[1 - 2 e^{-\kappa_2 m} \left(\frac{42e}{\delta}\right)^k\right]$, ← Success!
where κ_1 is arbitrary and $\kappa_2 = \frac{\delta^2}{24} - \frac{1}{\kappa_1}$.

Proof outline:

Needed t.s.t. $\boxed{(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2}$

$\forall x \in \Sigma_k \triangleq \{x \in \mathbb{R}^N, \|x\|_0 \leq k\}$, w.h.p.

- Chose a net of pts $Q_T \subset \mathcal{X}_T \triangleq \{x \in \mathbb{R}^N, \text{supp}(x) = T\}$ s.t. $\min_{q \in Q_T} \|x - q\|_2 \leq \frac{\delta}{14}$ for any $x \in \mathcal{X}_T$ with $\|x\|_2 = 1$.

  $Q \triangleq \bigcup_{T: |T| = k} Q_T$. Note: $\|q\|_2 = 1 \; \forall q \in Q_T$.

- Used Lemma 1 t.s.t.
  $(1-\varepsilon)\|q\|_2^2 \leq \|Aq\|_2^2 \leq (1+\varepsilon)\|q\|_2^2 \quad \forall q \in Q$
  w.p. $\geq 1 - 2 e^{-m C_0(\varepsilon)} \left(\frac{42eN}{\delta k}\right)^k$.

- Set $\boxed{\varepsilon = \frac{\delta}{\sqrt{2}}}$ and showed that the prob. can be further lower bdd as follows:
  $1 - 2e^{-m C_0(\varepsilon)}\left(\frac{42eN}{\delta k}\right)^k \geq 1 - 2\left(\frac{42e}{\delta}\right)^k e^{-m\left(\frac{\delta^2}{24} - \frac{1}{\kappa_1}\right)}$
  - Used the assump. that $\boxed{m \geq \kappa_1 k \log\left(\frac{N}{k}\right)}$.
  - We thus have the reqd. prob. over the net of pts.
    Needed to still s.t. A satisfies RIP with δ.

- Defined δ_k = smallest # s.t.
  $\|Ax\|_2 \leq \sqrt{1+\delta_k}\,\|x\|_2 \quad \forall x \in \Sigma_k, \|x\|_2 \leq 1$.

- Used Lemma 2 t.s.t. $\forall x \in \Sigma_k$, $\|x\|_2 = 1$,
  $\|Ax\|_2 \leq \sqrt{1 + \frac{\delta}{\sqrt{2}}} + \sqrt{1+\delta_k}\,\frac{\delta}{14}$

Since δ_k is the smallest # s.t.
$\|Ax\|_2 \leq \sqrt{1+\delta_k} \quad \forall x \in \Sigma_k, \|x\|_2 = 1$,
we must have
$\sqrt{1+\delta_k} \leq \sqrt{1 + \frac{\delta}{\sqrt{2}}} + \sqrt{1+\delta_k} \cdot \frac{\delta}{14}$.

$\Rightarrow \boxed{\delta_k \leq \delta}$.

Similarly, $\|Ax\|_2 \geq \sqrt{1 - \frac{\delta}{\sqrt{2}}} - \sqrt{1+\delta}\,\frac{\delta}{14} \geq \sqrt{1-\delta}$
$\forall x \in \Sigma_k$, $\|x\|_2 = 1$.

Thus, $\forall x \in \Sigma_k$, $(1-\delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|_2^2$
holds w.p. exceeding $1 - 2\left(\frac{42e}{\delta}\right)^k e^{-\kappa_2 m}$,
where $\kappa_2 = \frac{\delta^2}{24} - \frac{1}{\kappa_1}$, provided $m \geq \kappa_1 k \log\left(\frac{N}{k}\right)$. ∅

To finish the proof, need to prove the two Lemmas:

Lemma 1: $A \in \mathbb{R}^{m \times N}$, $A_{ij} \overset{iid}{\sim} \mathcal{N}(0, \frac{1}{m})$. Then, for any fixed $x \in \mathbb{R}^N$,
$$\Pr\left\{ \left| \|Ax\|_2^2 - \|x\|_2^2 \right| \geq \varepsilon \|x\|_2^2 \right\} \leq 2 e^{-m C_0(\varepsilon)}$$
$\forall\, 0 < \varepsilon < 1$, where $C_0(\varepsilon) = \frac{\varepsilon^2}{4} - \frac{\varepsilon^3}{6}$.
[$C_0(\varepsilon) = \frac{\varepsilon^2}{12}$ works too.]

Lemma 2: Let $\varepsilon \in (0,1)$. $\exists$ a set of pts $Q \in \mathbb{R}^k$ s.t. $\|q\|_2 = 1 \; \forall q \in Q$, $|Q| < \left(\frac{3}{\varepsilon}\right)^k$, and for any $x \in \mathbb{R}^k$, $\|x\|_2 = 1$, $\min_{q \in Q} \|x - q\|_2 \leq \varepsilon$.

Proof of Lemma 1:

Suffices t.s.t.
$$\begin{cases} \Pr\{\|Ax\|_2^2 \geq (1+\varepsilon)\|x\|_2^2\} \leq e^{-m C_0(\varepsilon)} \\ \Pr\{\|Ax\|_2^2 \leq (1-\varepsilon)\|x\|_2^2\} \leq e^{-m C_0(\varepsilon)} \end{cases}$$

$|a| \geq b$
$\{a \geq b\} \cup \{-a \geq b\}$

$\Pr\{\{\|Ax\|_2^2 - \|x\|_2^2 \geq \varepsilon\|x\|_2^2\} \cup \{\|x\|_2^2 - \|Ax\|_2^2 \geq \varepsilon\|x\|_2^2\}\}$
$\leq \Pr\{\|Ax\|_2^2 \geq (1+\varepsilon)\|x\|_2^2\} + \Pr\{\|Ax\|_2^2 \leq (1-\varepsilon)\|x\|_2^2\}$.
$\leq e^{-m C_0(\varepsilon)} + e^{-m C_0(\varepsilon)} = 2 e^{-m C_0(\varepsilon)}$

Suffices to consider x s.t. $\|x\|_2^2 = m$.
Let $v \triangleq Ax \in \mathbb{R}^m$. Then, $v_j \overset{iid}{\sim} \mathcal{N}(0,1)$

$\Rightarrow Y \triangleq \|v\|_2^2 = v_1^2 + \cdots + v_m^2$ is χ_m^2 distributed.

MGF: $\mathbb{E}\{e^{sY}\} = \frac{1}{(1-2s)^{m/2}}$, $s < \frac{1}{2}$.

Hence, $\Pr\{Y \geq (1+\varepsilon)m\} = \Pr\{e^{sY} \geq e^{s(1+\varepsilon)m}\}$
$\leq \frac{\mathbb{E}\{e^{sY}\}}{e^{s(1+\varepsilon)m}}$ (Markov Ineq.)
$= \frac{1}{(1-2s)^{m/2}\, e^{s(1+\varepsilon)m}}$.

Can min. RHS w.r.t. s, s.t. $0 < s < \frac{1}{2}$.

$\Rightarrow \max_{0 < s < \frac{1}{2}} \; s(1+\varepsilon)m + \frac{m}{2}\log(1-2s)$

$s_{opt} = \frac{1}{2}\left(\frac{\varepsilon}{1+\varepsilon}\right)$: satisfies $0 < s < \frac{1}{2}$.

$\Rightarrow \Pr\{Y \geq (1+\varepsilon)m\} \leq (1+\varepsilon)^{\frac{m}{2}} e^{-(1+\varepsilon)\left(\frac{1}{2}\right)\left(\frac{\varepsilon}{1+\varepsilon}\right)m}$
$= e^{\frac{m}{2}\log(1+\varepsilon)}\, e^{-\frac{m\varepsilon}{2}}$.

Now $\log(1+\varepsilon) \leq \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3}$ [Show!]

$\Rightarrow \Pr\{Y \geq (1+\varepsilon)m\} \leq e^{\frac{m\varepsilon}{2}} e^{-\frac{m}{2}\left(\frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3}\right)} e^{-\frac{m\varepsilon}{2}}$
$= e^{-\frac{m}{2}\left(\frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3}\right)} \leq e^{-\frac{m\varepsilon^2}{12}}$ (show!)
for $0 < \varepsilon < 1$

Proof for $\Pr\{Y \leq (1-\varepsilon)m\}$ is similar [HW].
Hence, $\Pr\{|Y - m| \geq \varepsilon m\} \leq 2e^{-\frac{m\varepsilon^2}{12}}$.

∅

Proof of Lemma 2

Constructive proof. Greedy procedure for constructing Q:
1. Pick $q_1 \in \mathbb{R}^k$ s.t. $\|q_1\|_2 = 1$.
2. At step i, pick any $q_i \in \mathbb{R}^k$ s.t. $\|q_i\|_2 = 1$ and $\|q_i - q_j\| > \varepsilon \;\; \forall j < i$.
   Add q_i to Q.
3. Repeat (2) till no more pts can be added.

Want to bound $|Q|$.

… of radius (ε) at each q_i

Center balls of radius $\left(\frac{\varepsilon}{2}\right)$ ...

These balls are disjoint, and all balls lie inside a "big" ball with radius $\left(1+\frac{\varepsilon}{2}\right)$.

Hence, $|Q| \, \underline{\mathrm{vol}\left(B_k\left(\frac{\varepsilon}{2}\right)\right)} \leq \underline{\mathrm{Vol}\left(B_k\left(1+\frac{\varepsilon}{2}\right)\right)}$

where $B_k(r)$ = Ball of radius r in $\mathbb{R}^k$.

Hence $|Q| \leq \dfrac{\left(1+\frac{\varepsilon}{2}\right)^k}{(\varepsilon/2)^k} \underset{(\varepsilon<1)}{\leq} \dfrac{(3/2)^k}{(\varepsilon/2)^k} = \left(\frac{3}{\varepsilon}\right)^k.$

$\square$

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A satisfies RIP w.p. $\geq 1 - 2e^{-\kappa_2 m}\left(\frac{42e}{\delta}\right)^k$,

where $\kappa_2 = \frac{\delta^2}{24} - \frac{1}{\kappa_1}$.

<u>Remark</u>: Often, we are interested in signals that are sparse (or compressible) in some other orthonormal basis $\Psi \neq I$.

$x = \underline{\Psi} \underset{\text{sparse}}{z}$.

$\Rightarrow A\Psi$ needs to satisfy RIP.

$Ax = (A\Psi)z$

Choose the net of pts in the k-dim. space spanned by sets of k cols of Ψ! Then the proof goes through!

$\Rightarrow$ All matrices of the form $\underline{A}\,\underline{\Psi}$ satisfy RIP!

Gaussian measurement matrices are <u>universal</u> for taking compressive measurements.

Random meas. matrices $\Rightarrow$ observations are "democratic", i.e., no one measurement is more imp. than others.

(In contrast, deterministic matrices do not share these two properties.)